DENSITÀ DI MASSA

$$\rho_{medio} = \frac{\Delta m}{\Delta V} \qquad \rho(\vec{r}) = \lim_{\Delta V \to 0} \rho_{medio} = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

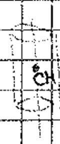
ogni volumetto dV ho dm

$$M = \sum_i m_i = \int_{corpo} dm = \int_{corpo} \rho \, dV$$

corpo omogeneo: ρ = costante nel corpo

se ho centro di simmetria

se ho l'asse di simmetria



$$\boxed{\vec{r}_{CM} = \frac{\sum_{i=1}^{n} (m_i \vec{r}_i)}{M}}$$

MOTO DEL CENTRO DI MASSA

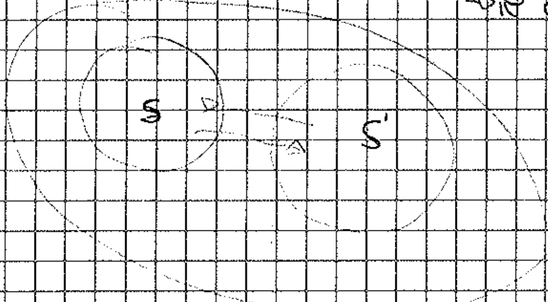
$$M = \sum_{i=1}^{n} m_i$$

$$\vec{P} = \sum_i m_i \vec{v}_i = M \vec{v}_{CM}$$

$$\vec{v}_{CM} \equiv \frac{\vec{P}}{M} = \frac{d}{dt} \vec{r}_{CM}$$

Per un sistema isolato: $\vec{P} = \text{cost}$

↳ il CM di un sistema isolato si muove con velocità costante



$S + S'$ isolato

particelle S : i

particelle S' : j

$$\vec{P}_{S+S'} = \text{cost} = \sum_{i \in S} \vec{P}_i + \sum_{j \in S'} \vec{P}_j$$

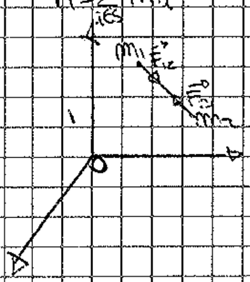
$$\Delta \vec{P}_{S+S'} = 0$$

$$\Delta \vec{P}_S = -\Delta \vec{P}_{S'}$$

$$\vec{F}^{S}_{est} = \frac{d\vec{P}_S}{dt} = -\frac{d\vec{P}_{S'}}{dt} = -\vec{F}^{S'}_{est}$$

$$\vec{P}^S = M \vec{v}_{CM} \qquad \vec{F}_{est} = \frac{d}{dt} \vec{P}^S = M \vec{a}_{CM}$$

$$M = \sum_{i \in S} m_i$$



SISTEMA ISOLATO

$$\vec{F}_{21} = -\vec{F}_{12}$$

$$\begin{cases} \vec{F}_{12} = m_1 \frac{d\vec{v}_1}{dt} \\ \vec{F}_{21} = m_2 \frac{d\vec{v}_2}{dt} \end{cases}$$

$$\frac{d}{dt} \vec{v}_1 - \frac{d}{dt} \vec{v}_2 = \frac{d}{dt} (\vec{v}_1 - \vec{v}_2) = \vec{a}_{12}$$

$$\hookrightarrow \frac{1}{m_1} \vec{F}_{12} - \frac{1}{m_2} \vec{F}_{21} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \vec{F}_{12}$$

MASSA RIDOTTA

$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}}$$

$$\boxed{\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}} \qquad \vec{F}_{12} = \mu \vec{a}_{12}$$

se $m_1 = m_2 \Rightarrow \mu = \frac{m^2}{2m} = \frac{1}{2} m$

se $m_1 \gg m_2 \Rightarrow \mu = \frac{m_1 m_2}{m_1 + m_2} = m_1$

- $\vec{r}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$

In un sistema di riferimento CM:

$$\begin{cases} \vec{r}_2\,' = \vec{r}_2 - \vec{r}_{CM} = -\frac{m_1}{m_1+m_2}(\vec{r}_1 - \vec{r}_2) \\ \vec{r}_1\,' = \vec{r}_1 - \vec{r}_{CM} = \frac{m_2}{m_1+m_2}(\vec{r}_1 - \vec{r}_2) \end{cases} \qquad \begin{cases} \vec{v}_1\,' = \frac{m_2}{m_1+m_2}\vec{v}_{12} \\ \vec{v}_2\,' = -\frac{m_1}{m_1+m_2}\vec{v}_{12} \end{cases}$$

$$\begin{cases} \vec{P}_1\,' = \frac{m_1 m_2}{m_1+m_2}\vec{v}_{12} = \mu \vec{v}_{12} \\ \vec{P}_2\,' = -\frac{m_1 m_2}{m_1+m_2}\vec{v}_{12} = -\mu \vec{v}_{12} \end{cases} \qquad \boxed{\vec{P}_1\,' + \vec{P}_2\,' = 0}$$

$$\vec{F}_{12} = \frac{d\vec{P}_1\,'}{dt} = \mu \vec{a}_{12}$$

Energia cinetica nel sistema del CM: $|\vec{P}_1\,'| = \mu|\vec{v}_{12}| = |\vec{P}_2\,'| = \vec{P}_{12}$

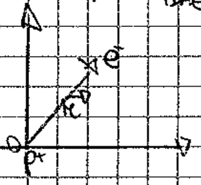
$$E_K' = \frac{(p_1')^2}{2m_1} + \frac{(p_2')^2}{2m_2} = \frac{P_{12}^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{P_{12}^2}{2\mu} = \frac{1}{2}\mu|\vec{v}_{12}|^2$$

Esempio:

atomo idrogeno: $\begin{matrix} p^+ \\ e^- \end{matrix} \begin{cases} m_p \sim 1.67 \times 10^{-27}\,kg \\ m_e \sim 9.11 \times 10^{-31}\,kg \end{cases}$

$M = m_e + m_p \simeq m_p$

$\mu = \frac{m_e m_p}{m_e + m_p} \simeq m_e$



$$\vec{r}_{CM} = \frac{m_e \vec{r} + m_p \vec{0}}{m_e + m_p} = \frac{m_e}{m_e + m_p}\vec{r} \sim \vec{0}$$

Nel sistema di riferimento CM: il protone è fermo in O

$$\vec{v}_{CM} = \frac{m_e \vec{v}_e + m_p \vec{v}_p}{m_e + m_p} \sim \vec{v}_p$$

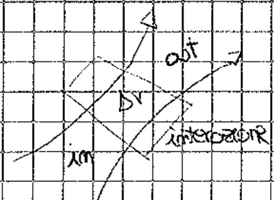
$$\begin{cases} \vec{v}_e\,' = \frac{m_p}{m_e+m_p}(\vec{v}_e - \vec{v}_p) \sim \vec{v}_e - \vec{v}_p \\ \vec{v}_p\,' = \frac{m_e}{m_e+m_p}(\vec{v}_p - \vec{v}_e) \sim 0 \end{cases}$$

$$\begin{cases} m_1, \ \vec{v}_1 \\ m_2, \ \vec{v}_2 \end{cases} \qquad \underline{E_K} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1|\vec{v}_1\,' + \vec{v}_{CM}|^2 + \frac{1}{2}m_2|\vec{v}_2\,' + \vec{v}_{CM}|^2 =$$

$$= \underbrace{\frac{1}{2}m_1(v_1')^2 + \frac{1}{2}m_2(v_2')^2} + \underbrace{\frac{1}{2}(m_1+m_2)v_{CM}^2} + \underline{m_1\vec{v}_1\,'\cdot\vec{v}_{CM} + m_2\vec{v}_2\,'\cdot\vec{v}_{CM}} =$$

$$= \underline{\frac{1}{2}\mu v_{12}^2 + \frac{1}{2}M v_{CM}^2} + \underbrace{(\vec{P}_1\,' + \vec{P}_2\,')\cdot\vec{v}_{CM}}_{=0}$$

URTI



Particelle in moto che, durante Δt breve, in regione limitata di spazio Δr, interagiscono e cambiano stato di moto.

$(\vec{P}_i \; E_{ki})$ in

$(\vec{P}_j \; E_{kj})$ out

Nell'urto entrano in gioco solo FORZE INTERNE ⇒ SISTEMA ISOLATO

⇒ negli urti la quantità di moto totale del sistema è conservata

$$\boxed{\vec{P}^{in} = \sum_{i=1}^{n} \vec{p}_i^{\,in} = \vec{P}^{out} = \sum_{j=1}^{m} \vec{p}_j^{\,out}} \rightarrow \sum_i \vec{p}_i^{\,in} = \sum_j \vec{p}_j^{\,out}$$

VINCOLI CINEMATICI

$$E_k^{in} = \sum_{i=1}^{n} \left(\frac{1}{2} \frac{(p_i^{in})^2}{m_i} \right)$$

$$E_k^{out} = \sum_{j=1}^{m} \left(\frac{1}{2} \frac{(p_j^{out})^2}{m_j} \right)$$

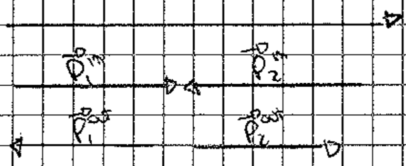
⇒ se $E_k^{out} = E_k^{in}$ URTO ELASTICO

se $E_k^{out} \neq E_k^{in}$ URTO ANELASTICO

$$Q = E_k^{out} - E_k^{in} \rightarrow = 0 \text{ urto elastico}$$

coefficiente di anelasticità

URTI IN UNA DIMENSIONE



2 casi limite:

ⓐ URTO ELASTICO

$$\begin{cases} p_1^{in} + p_2^{in} = p_1^{out} + p_2^{out} \\ \dfrac{(p_1^{in})^2}{2m_1} + \dfrac{(p_2^{in})^2}{2m_2} = \dfrac{(p_1^{out})^2}{2m_1} + \dfrac{(p_2^{out})^2}{2m_2} \end{cases} \qquad \begin{cases} m_1 (v_1^{in} - v_1^{out}) = m_2 (v_2^{out} - v_2^{in}) \\ \frac{1}{2} m_1 \left[(v_1^{in})^2 - (v_1^{out})^2 \right] = \frac{1}{2} m_2 \left[(v_2^{out})^2 - (v_2^{in})^2 \right] \end{cases}$$

$$\begin{cases} m_1 (v_1^{in} - v_1^{out}) = m_2 (v_2^{out} - v_2^{in}) \\ \frac{1}{2} m_1 (v_1^{in} - v_1^{out})(v_1^{in} + v_1^{out}) = \frac{1}{2} m_2 (v_2^{out} - v_2^{in})(v_2^{out} + v_2^{in}) \end{cases}$$

$$\begin{cases} v_1^{in} + v_1^{out} = v_2^{in} + v_2^{out} \\ v_1^{in} - v_1^{out} = \frac{m_2}{m_1} (v_2^{out} - v_2^{in}) \end{cases} \qquad \begin{cases} v_1^{out} = \frac{m_1 - m_2}{m_1 + m_2} v_1^{in} + \frac{2m_2}{m_1 + m_2} v_2^{in} \\ v_2^{out} = \frac{2m_1}{m_1 + m_2} v_1^{in} + \frac{m_2 - m_1}{m_1 + m_2} v_2^{in} \end{cases}$$

casi particolari:

- $m_1 = m_2$ $\quad \begin{cases} v_1^{out} = v_2^{in} \\ v_2^{out} = v_1^{in} \end{cases}$

- $m_1 >> m_2$

  $m_1 + m_2 \approx m_1 - m_2 \approx m_1$ $\quad \begin{cases} v_1^{out} \approx v_1^{in} \\ v_2^{out} \approx 2v_1^{in} - v_2^{in} \end{cases}$

  $\frac{m_2}{m_1 + m_2} \approx 0$

- $v_2^{in} = 0$
  - $m_1 = m_2$ $\quad \begin{cases} v_1^{out} = 0 \\ v_2^{out} = v_1^{in} \end{cases}$
  - $m_1 >> m_2$ $\quad \begin{cases} v_1^{out} \approx v_1^{in} \\ v_2^{out} \approx 2v_1^{in} \end{cases}$
  - $m_2 >> m_1$ $\quad \begin{cases} v_1^{out} \approx -v_1^{in} \\ v_2^{out} \approx 0 \end{cases}$

conservazione cons $\vec{p}$: $\vec{p}^{\,in} = \vec{p}^{\,out}$

$$p_1^{in} + p_2^{in} = p^{in} = p^{out} = p_1^{out} + p_2^{out}$$

conservazione cons E_k:
$$\left(\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}\right)^{in} = \left(\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}\right)^{out}$$

{2 eq. in 2 incognite p_1^{out}, p_2^{out}}

(b) URTO COMPLETAMENTE ANELASTICO

v_1^{in}, v_2^{in}

$m_1 \rightarrow$ $\quad m_2 \quad$ $m_1+m_2 \xrightarrow{v^{out}}$

$$P = (m_1+m_2)\, v_{CM}$$

Consideriamo la conservazione quantità di moto:

$$m_1 v_1^{in} + m_2 v_2^{in} = (m_1+m_2)\, v^{out} \implies v^{out} = \frac{m_1 v_1^{in} + m_2 v_2^{in}}{m_1+m_2} = v_{CM}$$

$$Q = E_k^{out} - E_k^{in} = \frac{1}{2}(m_1+m_2)(v^{out})^2 - \frac{1}{2}m_1 v_1^{in^2} - \frac{1}{2}m_2 v_2^{in^2}$$
$$= \frac{1}{2}\cancel{(m_1+m_2)}\,\frac{m_1^2 v_1^{in^2} + m_2^2 v_2^{in^2} + 2m_1m_2 v_1^{in} v_2^{in}}{(m_1+m_2)^{\cancel{2}}} - \frac{1}{2}m_1 v_1^{in^2} - \frac{1}{2}m_2 v_2^{in^2} =$$
$$= \frac{1}{2(m_1+m_2)}\left[m_1^2 v_1^{in^2} + m_2^2 v_2^{in^2} + 2m_1m_2 v_1^{in} v_2^{in} - (m_1^2 + m_1m_2)v_1^{in^2} - (m_2^2 + m_1m_2)v_2^{in^2}\right] =$$
$$= \frac{m_1m_2}{2(m_1+m_2)}\left[2v_1^{in}v_2^{in} - v_1^{in^2} - v_2^{in^2}\right] = -\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(v_1^{in} - v_2^{in})^2$$

$$\boxed{Q = -\frac{1}{2}\mu v_{12}^2 < 0} \qquad \implies E_k^{out} < E_k^{in}$$

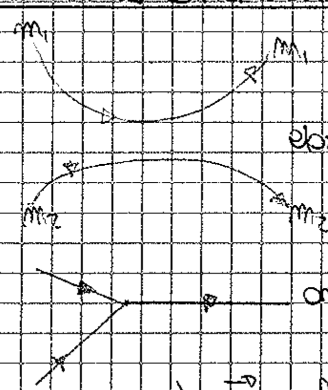
$\mu = \dfrac{m_1m_2}{m_1+m_2}$ massa ridotta

$$E_k = E_k^{int} + \frac{1}{2}M v_{CM}^2 \qquad \boxed{E_k^{int} = \frac{1}{2}\mu v_{12}^2}$$

$$E_k^{in} = \frac{1}{2}\mu v_{12}^2 + \frac{1}{2}M(v_{CM}^2)^{in} \qquad E_k^{out} = Q + E_k^{in} = E_k^{in} - \frac{1}{2}\mu v_{12}^2$$

$\implies$ nell'urto tutta E_k^{int} è stata dissipata

URTI IN 2 DIMENSIONI



elastico

anelastico

$$\implies \vec{v}_{out} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2} = \vec{v}_{CM}$$

(b) urto completamente anelastico

$$\vec{v}^{\,out} = (v_x^{out}, v_y^{out})$$

Conservazione quantità di moto: $m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{v}_{out}$

$$\hookrightarrow \begin{cases} m_1 v_{1x} + m_2 v_{2x} = (m_1+m_2)\, v_x^{out} \\ m_1 v_{1y} + m_2 v_{2y} = (m_1+m_2)\, v_y^{out} \end{cases}$$

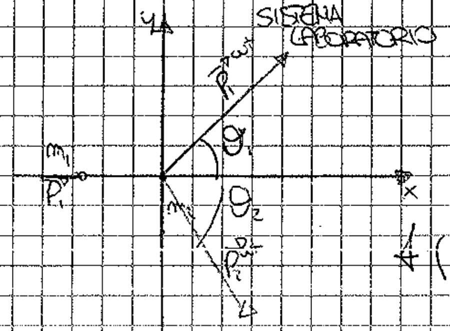
(a) urto elastico

$$\vec{p}_1^{\,in}, \vec{p}_2^{\,in} \longrightarrow (\vec{p}_1^{\,out}, \vec{p}_2^{\,out})\,?\,(p_{1x}^{out}, p_{2x}^{out}, p_{1y}^{out}, p_{2y}^{out})$$

conservazione $\vec{p}$ $\begin{cases} p_x^{in} = p_x^{out} \\ p_y^{in} = p_y^{out} \end{cases}$ $p_{1x}^{in} + p_{2x}^{in} = p_{1x}^{out} + p_{2x}^{out}$

$p^2 = |\vec{p}|^2 = \vec{p} \cdot \vec{p}$

conservazione E_k $\frac{(p_1^{in})^2}{2m_1} + \frac{(p_2^{in})^2}{2m_2} = \frac{(p_1^{out})^2}{2m_1} + \frac{(p_2^{out})^2}{2m_2}$



$\vec{p}_1 = (p_1, \theta_1)$

$\vec{p}_2 = (p_2, \theta_2)$

$\forall (\theta, \vec{p}_1^{in})$ $p_1^{out} = p_1^{out}(\theta, \vec{p}^{in})$

$p_2^{out}(\theta_1, \vec{p}^{in})$

$\vec{p}_1^{in}, \vec{p}_2^{in} = 0$ $\theta_2(\theta_1, \vec{p}_1^{in})$

$m_1 = m_2$

$\begin{cases} \vec{p}^{in} = \vec{p}_1^{out} + \vec{p}_2^{out} \\ (p_1^{in})^2 = (p_1^{out})^2 + (p_2^{out})^2 \end{cases}$ $\Rightarrow$ $\vec{p}_1^{in} \cdot \vec{p}_1^{in} = (\vec{p}_1^{out} + \vec{p}_2^{out}) \cdot (\vec{p}_1^{out} + \vec{p}_2^{out})$

$(p_1^{in})^2 = (p_1^{out})^2 + (p_2^{out})^2 + 2\vec{p}_1^{out} \cdot \vec{p}_2^{out}$

$\theta_1 + \theta_2 = \frac{\pi}{2}$ $\Rightarrow$ $\boxed{\vec{p}_1^{out} \cdot \vec{p}_2^{out} = 0}$ nel sistema laboratorio per $\vec{p}_2^{in} = 0$

SISTEMA RIFERIMENTO CM

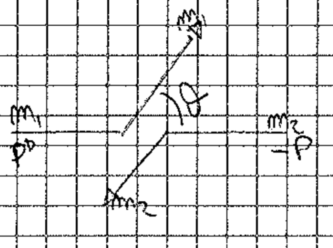
$\vec{v}_{CM}^{in} = \frac{\vec{p}_1^{in} + \overbrace{\vec{p}_2^{in}}^{=0}}{m_1 + m_2} = \frac{m_1}{m_1 + m_2} \vec{v}_1^{in}$

$\vec{v}_1'^{in} = \vec{v}_1^{in} - \vec{v}_{CM} = \frac{m_2}{m_1 + m_2} \vec{v}_1^{in}$

$\vec{v}_2'^{in} = \cancel{\vec{v}_2^{in}} - \vec{v}_{CM} = -\frac{m_1}{m_1 + m_2} \vec{v}_1^{in}$

$\begin{cases} \vec{p}_1'^{in} = m_1 \vec{v}_1' = \mu \vec{v}_1 \\ \vec{p}_2'^{in} = m_2 \vec{v}_2' = -\mu \vec{v}_1^{in} \end{cases}$

$\boxed{\vec{p}_1' + \vec{p}_2' = 0}$



in: $\vec{p}_1' = -\vec{p}_2' = \vec{p}$

conservazione quantità di moto

$\vec{p}^{in} = 0 \Rightarrow \vec{p}^{out} = 0$

$\vec{p}_1'^{out} = -\vec{p}_2'^{out} = \vec{q}$

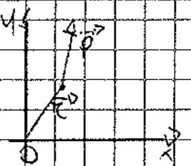
conservazione E_k (per urto elastico)

in $E_k^{in} = \frac{p^2}{2m_1} + \frac{p^2}{2m_2} = \frac{p^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{p^2}{2\mu}$

$E_k^{out} = \frac{q^2}{2\mu}$ $\frac{p^2}{2\mu} = \frac{q^2}{2\mu} \Rightarrow p = q$

SISTEMI DI PARTICELLE IN MOTO ROTAZIONALE

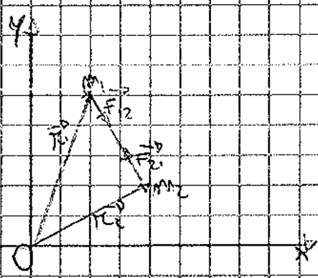
MOMENTO ANGOLARE: $\vec{L} = \vec{r} \times \vec{p}$
rispetto ad un punto (O)



MOMENTO MECCANICO DI UNA FORZA APPLICATA: $\vec{\tau} = \vec{r} \times \vec{F}$
$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Consideriamo sistema di 2 particelle interagenti: $\vec{F}_{12} = -\vec{F}_{21}$

$$\begin{cases} \vec{\tau}_1 = \frac{d}{dt}\vec{L}_1 \\ \vec{\tau}_2 = \frac{d}{dt}\vec{L}_2 \end{cases}$$ rispetto allo stesso punto O



sistema non isolato: su particella 1 $\vec{F}_{12} + \vec{F}_1$

su particella 2 $\vec{F}_{21} + \vec{F}_2$

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = \frac{d}{dt}(\vec{L}_1 + \vec{L}_2)$$

$$\vec{\tau}_1 = \vec{r}_1 \times (\vec{F}_{12} + \vec{F}_1)$$

$$\vec{\tau}_2 = \vec{r}_2 \times (\vec{F}_{21} + \vec{F}_2)$$

$\vec{F}_{21} = -\vec{F}_{12}$

$$\Rightarrow \vec{\tau} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} + \overbrace{\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2}^{\vec{\tau}_{ext}}$$

calcoliamo $\underbrace{\vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}}_{=0}$ hanno direzioni opposte

$$|\vec{r}_1 \times \vec{F}_{12}| = r_1 F_{12} \sin\theta$$

$$r_1 \sin\theta_1 = OH = r_2 \sin\theta_2$$

$$|\vec{r}_2 \times \vec{F}_{21}| = r_2 |\vec{F}_{21}| \sin\theta_2 = \overline{OH}\, |F_{12}|$$

$$|\vec{r}_1 \times \vec{F}_{12}| = |\vec{r}_2 \times \vec{F}_{21}|$$ direzione opposta

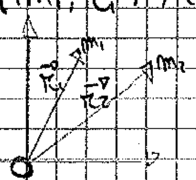
$$\boxed{\frac{d}{dt}(\vec{L}_1 + \vec{L}_2) = \vec{\tau}_{ext}} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$

$$\frac{d}{dt}\vec{P} = \vec{F}_{ext}$$ → sistema di particelle $\vec{L}_i$ $i = 1, \dots, n$

$$\vec{L} = \sum_{i=1}^{n} \vec{L}_i = \vec{L}_1 + \vec{L}_2 + \dots + \vec{L}_n$$

$$\frac{d}{dt}\vec{L} = \vec{\tau}_{ext}$$

$(m_1, \vec{r}_1), (m_2, \vec{r}_2)$



$\vec{L}_c$? $\vec{L}$ rispetto al CM del sistema

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

Posizione particelle risp. CM
$$\begin{cases} \vec{r}_1{}' = \vec{r}_1 - \vec{r}_{CM} = \frac{m_2}{m_1+m_2}\vec{r}_{12} \\ \vec{r}_2{}' = \vec{r}_2 - \vec{r}_{CM} = -\frac{m_1}{m_1+m_2}\vec{r}_{12} \end{cases} \qquad \begin{cases} \vec{p}_1{}' = \mu\vec{v}_{12} \\ \vec{p}_2{}' = -\mu\vec{v}_{12} \end{cases}$$

$$\vec{L}_c = \vec{r}_1{}' \times \vec{p}_1{}' + \vec{r}_2{}' \times \vec{p}_2{}' = \left(\frac{m_2}{m_1+m_2}\vec{r}_{12}\right) \times (\mu\vec{v}_{12}) + \left(-\frac{m_1}{m_1+m_2}\vec{r}_{12}\right) \times (-\mu\vec{v}_{12})$$
$$= \vec{r}_{12} \times \vec{v}_{12}\left(\frac{m_2+m_1}{m_1+m_2}\right)\mu = \boxed{\mu\vec{r}_{12} \times \vec{v}_{12} = \vec{L}_c}$$

Vediamo come è legato a $\vec{L}$ sistema riferimento laboratorio

$$\begin{cases} \vec{p}_1 = \vec{p}_1{}' + m_1\vec{v}_{CM} \\ \vec{p}_2 = \vec{p}_2{}' + m_2\vec{v}_{CM} \end{cases}$$

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = (\vec{r}_1' + \vec{r}_{CM}) \times (\vec{p}_1' + m_1 \vec{v}_{CM}) + (\vec{r}_2' + \vec{r}_{CM}) \times (\vec{p}_2' + m_2 \vec{v}_{CM})$$

$$= \underbrace{\left[\vec{r}_1' \times \vec{p}_1' + \vec{r}_2' \times \vec{p}_2'\right]}_{\vec{L}_c} + \vec{r}_{CM} \times \left[\underbrace{\vec{p}_1' + \vec{p}_2'}_{=0} + \underbrace{(m_1 + m_2)\vec{v}_{CM}}_{\vec{P}}\right] + (m_1 \vec{r}_1' + m_2 \vec{r}_2') \times \vec{v}_{CM}$$

$$\boxed{\vec{L} = \vec{L}_c + \vec{r}_{CM} \times \vec{P}}$$

$$\vec{L} = \vec{L}_c + \vec{r}_{CM} \times \vec{P} \qquad \frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$$

$$\vec{\tau}_{ext} = \frac{d}{dt}\vec{L}_c + \underbrace{\vec{v}_{CM} \times \vec{P}}_{=0} + \vec{r}_{CM} \times \frac{d\vec{P}}{dt} = \frac{d\vec{L}_c}{dt} + \vec{r}_{CM} \times \frac{d\vec{P}}{dt}$$

$$\vec{v}_{CM} = \frac{\vec{P}}{M}$$

$$\vec{r} = \vec{r}' + \vec{r}_{CM}$$

$$\vec{\tau}_{ext} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots = (\vec{r}_1' + \vec{r}_{CM}) \times \vec{F}_1 + (\vec{r}_2' + \vec{r}_{CM}) \times \vec{F}_2$$

$$\vec{\tau}_{ext} = \underbrace{\vec{r}_1' \times \vec{F}_1 + \vec{r}_2' \times \vec{F}_2}_{\vec{\tau}_c} + \vec{r}_{CM} \times \underbrace{(\vec{F}_1 + \vec{F}_2)}_{\vec{F}_{ext}}$$

$$\vec{\tau}_{ext} = \vec{\tau}_c + \vec{r}_{CM} \times \vec{F}_{ext} \qquad \vec{\tau}_c = \frac{d}{dt}\vec{L}_c$$

$$\vec{\tau}_{ext} = \frac{d\vec{L}_c}{dt} + \vec{r}_{CM} \times \frac{d\vec{P}}{dt} \qquad \vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

$$\begin{cases} \vec{P} = M \vec{v}_{CM} \\ \vec{F}_{ext} = \sum_i \vec{F}_i^{ext} \\ \to \dfrac{d\vec{P}}{dt} = \vec{F}_{ext} \end{cases} \qquad \begin{cases} \vec{L} = \vec{L}_c + \vec{r}_{CM} \times \vec{P} \\ \vec{\tau}_{ext} = \sum_i (\vec{r}_i \times \vec{F}_i) = \vec{\tau}_c + \vec{r}_{CM} \times \vec{F}_{ext} \\ \to \dfrac{d}{dt}\vec{L} = \vec{\tau}_{ext} \end{cases}$$

- $\vec{\tau}_{ext} = 0$ → non agiscono $\vec{F}_{ext}$ (forze esterne)
  → $\vec{F}_{ext}$ hanno momento nullo
  → $\sum_i \vec{\tau}_i = 0$

Se $\vec{\tau}_{ext} = 0$ → $\vec{L}$ è COSTANTE

$S + S' \Rightarrow \vec{L}_S + \vec{L}_{S'} =$ costante $\Rightarrow \Delta\vec{L}_S = -\Delta\vec{L}_{S'}$

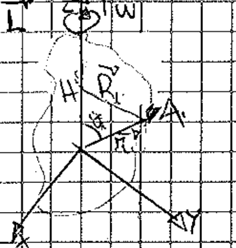
CORPO RIGIDO

Un sistema di particelle in cui le distanze tra le particelle sono costanti durante l'applicazione di forze o momenti.

- TRASLAZIONE: tutte le particelle descrivono traiettorie parallele
- ROTAZIONE intorno a un asse: tutte le particelle descrivono traiettorie circolari intorno ad una retta (asse di rotazione)

MOTO DEL CM: $\vec{F}_{ext} = M \frac{d\vec{v}_{CM}}{dt}$

MOTO ROTATORIO: → ASSE passa per il CM del corpo rigido
↘ ASSE passa per il generico punto fisso di un sistema inerziale



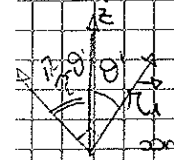
Ogni particella descrive un'orbita circolare, con centro sull'asse di rotazione e velocità angolare $\vec{\omega}$

A_i descrive un cerchio di raggio $R_i = r_i \sin\theta_i$

con velocità $\vec{v}_i = \vec{\omega} \times \vec{r}_i$

$v_i = \omega r_i \sin\theta_i = \omega R_i$

$\vec{L}_i = m_i \vec{r}_i \times \vec{v}_i$



$\vec{L} \perp \vec{r}_i, v_i \qquad \vec{r}_i \perp \vec{v}_i$

componente z: $L_{iz} = m_i r_i v_i \cos\left(\frac{\pi}{2} - \theta\right) = m_i \underbrace{r_i \sin\theta_i}_{R_i} \overset{\omega R_i}{v_i}$

$$\boxed{L_{iz} = m_i R_i^2 \omega}$$

$L_z = L_{1z} + L_{2z} + \ldots = \sum_i L_{iz} = m_1 R_1^2 + m_2 R_2^2 \omega + \ldots$

$L_z = \left(\sum_i m_i R_i^2\right)\omega$

$\sum_i m_i R_i^2 = I$ momento di inerzia $\qquad [I] = m\, \ell^2 \quad SI \quad kg\, m^2$

$L_z = I\omega \qquad \vec{P} = M \vec{v}_{CM}$

$\vec{L} \not\parallel \vec{\omega}$ in generale $\vec{L}$ non è parallelo all'asse di rotazione

per ogni corpo $\exists$ sempre 3 assi mutuamente $\perp$ tali che, se il corpo ruota intorno a uno di essi $\Rightarrow \vec{L} \parallel \vec{\omega}$. In questo caso $\vec{L} = I\vec{\omega}$ ASSI PRINCIPALI DI INERZIA x_0, y_0, z_0

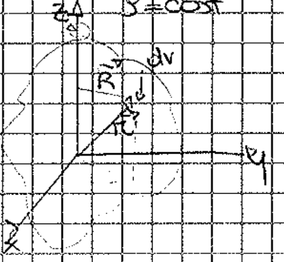
Gli assi principali si incontrano ~~[illegible]~~ nel CM del corpo

corpo omogeneo e ho ~~[illegible]~~ un asse di simmetria $\rightarrow$ coincide con uno degli assi principali

$m_i \rightarrow dm = \rho\, dV \qquad \rho = cost$

$I = \int_V dm R^2 = \int_V \rho R^2 dV$

$I = \int \rho R^2 dV$



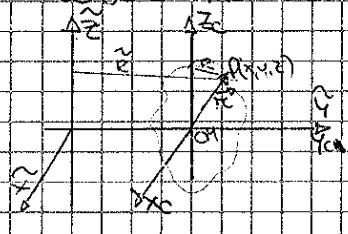
$R^2 = x^2 + y^2$

$I_z = \int \rho (x^2 + y^2) dV$

$I_y = \int \rho (x^2 + z^2) dV$

$I_x = \int \rho (y^2 + z^2) dV$

Consideriamo 2 assi //: come sono collegati I_{z_c} e $I_{\tilde{z}}$



TEOREMA DI STEINER in SRC $P(x, y, z)$

$\overline{OC} = a$

$R^2 = x^2 + y^2$

$\tilde{R}^2 = x^2 + (y + a)^2 = x^2 + y^2 + 2ay + a^2$

$I_{z_c} = \int \rho R^2 dV$

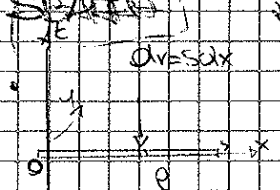
$I_{\tilde{z}} = \int \rho \tilde{R}^2 dV = \int \rho (R^2 + 2ay + a^2) dV$

$= I_{z_c} + a^2 \underbrace{\int \rho\, dV}_{m = M} + \underbrace{2a \int \rho y\, dV}_{= 0}$

$\int \rho y\, dV = \sum_i m_i y_i = M y_{CM}$

$$\boxed{I = I_c + Ma^2}$$

SBARRA



$dV = S\,dx$

$S \ll \ell$

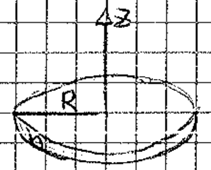
$I_0 = \int_V \rho (x^2 + y^2) dV \sim \int_V \rho x^2 dV = \rho \int_0^\ell x^2 S\, dx = \rho S \int_0^\ell x^2 dx = \frac{1}{3} \rho S \ell^3 = \frac{1}{3} (\rho V) \ell^2 = \frac{1}{3} M \ell^2$

Rispetto a CM

$-\frac{\ell}{2} \le x \le \frac{\ell}{2}$ $\quad I_c = \rho \int_{-\ell/2}^{\ell/2} x^2 S\,dx = \left[\rho S \frac{1}{3} x^3\right]_{-\ell/2}^{\ell/2} = \rho \frac{S\ell^3}{12} = \frac{1}{12} M\ell^2$

$$\left[I_c = \frac{1}{3}M\ell^2 - M\frac{\ell^2}{4} = \frac{1}{12} M\ell^2 \right]$$

DISCO OMOGENEO



$h \ll R$ coordinate cilindriche

$I_z =$

$dV = h \cdot 2\pi r\, dr$

$I_z = \int_V \rho r^2 dV = \int_0^R \rho h 2\pi r\, r^2 dr =$

$= 2\pi \rho h \int_0^R r^3 dr = 2\pi \rho h \frac{1}{4} R^4 = \frac{1}{2} M R^2$

$I_x = \int \rho\, dV (y^2 + z^2)$ $\quad h \ll R$

$dV = h\, dr\, r\, d\varphi$ $\quad = \rho \int y^2 dV \longrightarrow \rho \int_0^{2\pi} \int_0^R r^2 \sin^2\varphi\, h r\, d\varphi\, dr$

$0 \le r \le R$
$0 \le \varphi \le 2\pi$

$y = r \sin\varphi$

$I_x = \left[\underbrace{\int_0^{2\pi} d\varphi \sin^2\varphi}_{\pi} \left[\underbrace{\int_0^R dr\, r^3}_{R^4/4}\right] \rho h\right] = \frac{1}{4} \pi R^4 \rho h = \frac{1}{4} M R^2$

$I_x \approx \int \rho y^2 dV$

$I_y \approx \int \rho x^2 dV = I_x$

$I_z \approx \int \rho x^2 dV + \int \rho y^2 dV = I_y + I_x = 2 I_x$

$I_x = \frac{1}{2} I_z = \frac{1}{4} M R^2$

EQUAZIONI DEL MOTO ROTATORIO DI UN CORPO RIGIDO

$\vec{L} = \sum_i \vec{L}_i$ rispetto a O

$\frac{d\vec{L}}{dt} = \vec{\tau}$ rispetto a O

Supponiamo che l'asse di rotazione contenga O

a) z è principale $\quad \vec{L} = I\vec{\omega} \quad \vec{\tau} = \frac{d}{dt}(I\vec{\omega}) = \frac{dI}{dt}\vec{\omega} + I \frac{d\vec{\omega}}{dt}$

Se l'asse z resta fisso rispetto al corpo $\Rightarrow \frac{dI}{dt} = 0$

$\vec{\tau} = I \frac{d\vec{\omega}}{dt} = I\vec{\alpha}$ $\quad I \longleftrightarrow m \quad \vec{\omega} \longleftrightarrow \vec{v} \quad \vec{\alpha} \longleftrightarrow \vec{a}$ $\quad \vec{L} \longleftrightarrow \vec{p} \quad \vec{\tau} \longleftrightarrow \vec{F}$

Quando $\vec{\tau} = 0 \Rightarrow I\vec{\omega} =$ costante

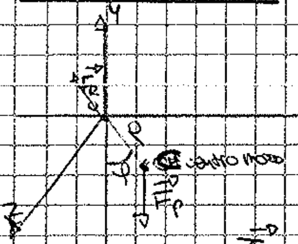
b) Quando z non è principale

$$L_z = I\omega \qquad \tau_z = \frac{d}{dt}(I\omega) \overset{\text{se } I \text{ costante}}{=} I\alpha$$

c) Quando l'asse di rotazione non ha punti fissi in s.r. inerziale

$$\tau_c = \frac{d}{dt}\vec{L}_c$$

PENDOLO FISICO



H

$\vec{F}_p = -Mg\,\vec{u}_y$

$\vec{F}_R$ forza di reazione vincolante

$\vec{\tau}_R = 0$

$\vec{\tau}_p = \vec{OC} \times \vec{F}_p = -Mg\ell \sin\varphi\, \vec{u}_z$

$\tau_z = I\alpha = I\frac{d^2\varphi}{dt^2}$ → $-Mg\ell\sin\varphi = I\frac{d^2\varphi}{dt^2}$; $\frac{Mg\ell}{I} = \omega_0^2$ $\quad \omega_0 = \sqrt{\frac{Mg\ell}{I}}$

$\alpha = \frac{d^2\varphi}{dt^2}$ acc. angolare

$$\boxed{\frac{d^2\varphi}{dt^2} + \omega_0^2 \sin\varphi = 0}$$

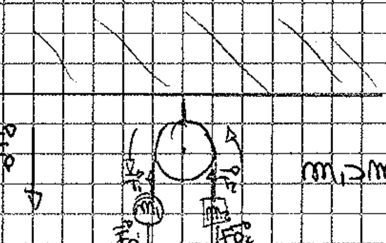
MOTO ARMONICO SEMPLICE

per piccole oscillazioni ($\sin\varphi \sim \varphi$): $\varphi = A\cos(\omega_0 t + \varphi_0) =$
$= A'\cos\omega_0 t + B\sin\omega_0 t$

$$\boxed{\frac{d^2\varphi}{dt^2} + \omega_0^2 \varphi = 0}$$

Per PENDOLO SEMPLICE

$\omega_0 = \sqrt{\frac{g}{\ell}}$



* se la puleggia è ideale: massa e attrito trascurabile

$m_1 > m_2$ → $a = \frac{m_1 - m_2}{m_1 + m_2} g$, $\quad T = \frac{2m_1 m_2}{m_1 + m_2} g \qquad T_1 = T_2 = T$

* se la puleggia non è ideale: $M \neq 0$ raggio R, superficie di contatto con la corda ha attrito (non scivola)

Trascuriamo il momento della forza d'attrito → ω antiorario

$m_1: \quad m_1 \cdot a = m_1 g - T_1$

$m_2: \quad m_2 a = T_2 - m_2 g$

puleggia: $\vec{\tau}_1 = (-R\vec{u}_y) \times (-T_1 \vec{u}_z) = RT_1 \vec{u}_x \qquad \vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$

$\vec{\tau}_2 = (R\vec{u}_y) \times (T_2 \vec{u}_z) = -RT_2 \vec{u}_x$

$T_1 = m_1(g - a) \qquad a = \alpha R$

$T_2 = m_2(g + a) \qquad I = \frac{1}{2}MR^2$ se disco sottile

$I\alpha = R(T_1 - T_2)$

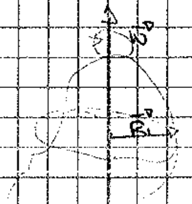
$\frac{1}{2}MR^{\cancel{2}}\frac{a}{\cancel{R}} = \cancel{R}(T_1 - T_2)$ → $\frac{1}{2}Ma = T_1 - T_2$

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}M}\, g$$

$$T_1 = \frac{2m_1 m_2 + \frac{1}{2} M m_2}{m_1 + m_2 + \frac{1}{2}M}\, g \qquad T_2 = \frac{2m_1 m_2 + \frac{1}{2} M m_1}{m_1 + m_2 + \frac{1}{2}M}\, g$$

ENERGIA CINETICA

$E_K = \frac{1}{2} \sum_i (m_i v_i^2)$



$v_i = \omega R_i$

Per corpo rigido $v_i = \omega R_i$

$$E_K = \frac{1}{2} \underbrace{\sum_i (m_i R_i^2)}_{I} \omega^2 = \frac{1}{2} I \omega^2$$

$\omega^2 = \vec\omega \cdot \vec\omega$

se l'asse è principale $\vec L = I \vec\omega$ $\quad L^2 = \vec L \cdot \vec L = I^2 \omega^2$

$$E_K = \frac{1}{2} I \frac{L^2}{I^2} = \frac{L^2}{2I}$$

ROTO-TRASLAZIONE

$E_K = E_{Kc} + \frac{1}{2} M v_c^2$

energia cinetica intorno rispetto al centro di massa

Se in sist. riferimento CM $\to E_{Kc} = \frac{1}{2} I_c \omega^2$

$E_K - E_{K0} = W_{ext}$

se Fext conservative $W_{ext} = E_p^{ext} - E_p^{ext}$

$\Rightarrow E_K + E_p^{ext} = E$

ROTAZIONE INTORNO ALL'ASSE PRINCIPALE

$\vec\tau = \frac{d\vec L}{dt}$ $\quad d\vec L = \vec\tau\, dt$ $\quad$ il momento angolare ruota nella direzione del momento applicato

Se $|\vec L| = cost$ $\quad \vec L \cdot \vec L = costante$

$$d\vec L \cdot \vec L + \vec L \cdot d\vec L = 0 = 2 \vec L\, d\vec L \Rightarrow d\vec L \perp \vec L$$

$$\vec\tau \perp \vec L$$

$$dt\, \vec\tau = d\vec L \Rightarrow d\vec L \perp \vec L \Rightarrow d\vec L \cdot \vec L = 0$$

$$d(L^2) = 0 \to L^2 = costante$$

nel moto circolare uniforme $|\vec v| = cost$

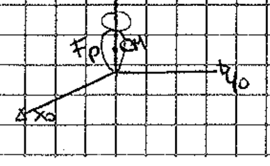
dir $\vec v$ ruota $\Rightarrow \exists F$

PRECESSIONE

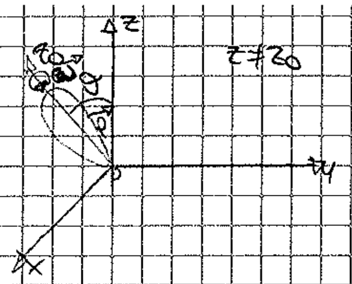
TROTTOLA

Corpo rigido che ruota intorno ad un asse principale

$\vec L = I \vec\omega$ $\quad \vec L, \vec\omega \parallel z_0$



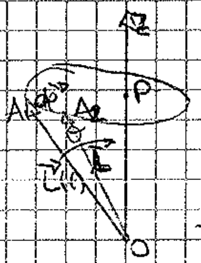
In questo caso F_P non ha momento



$z \neq z_0$ $\qquad \vec{\tau} = \vec{\ell} \times \vec{F}_p \qquad |\vec{\tau}| = Mg\ell \sin\theta$

$$\vec{\ell} \parallel z_0 \implies \vec{\tau} \perp z_0 \implies \vec{\tau} \perp \vec{L}$$

$$|\vec{L}| = \text{cost} \quad \text{ma} \quad d\vec{L} = \vec{\tau}\, dt (\perp \vec{L}) \neq 0$$



$$\overline{A_1P} = L\sin\theta = A_2P$$

$$\overline{A_1A_2} = dL$$

$t_2 - t_1 = dt \qquad d\varphi = A_1\hat{P}A_2 = \frac{\overline{A_1A_2}}{\overline{A_1P}} = \frac{dL}{L\sin\theta} = \frac{\tau\, dt}{L\sin\theta}$

La velocità angolare con cui $\vec{L}$ precede è

$$\Omega = \frac{d\varphi}{dt} = \frac{\tau}{L\sin\theta} = \frac{Mg\ell \cancel{\sin\theta}}{L\cancel{\sin\theta}} = \frac{Mg\ell}{L} \qquad [\Omega] = \frac{m\ell t^{-2}\ell}{m\ell^2 t^{-1}} = t^{-1}$$

Dovuto a precessione $\quad \vec{L} = I(\vec{\omega} + \vec{\Omega}) \qquad \underline{\Omega \ll \omega}$

$\omega \sim \Omega$

$\rightarrow$ fenomeno di NUTAZIONE (trottola)

<u>TERRA</u>

· le piano dell'ellittica (piano di rivoluzione intorno al sole)